Exercises for Section 4.1

1. Determine a condition on $|x-1|$ that will assure that:
   (a) $\left|x^2-1\right| < \frac{1}{2}$, (b) $\left|x^2-1\right| < 1/10^{-3}$,
   (c) $\left|x^2-1\right| < 1/n$ for a given $n \in \mathbb{N}$, (d) $\left|x^3-1\right| < 1/n$ for a given $n \in \mathbb{N}$.
2. Determine a condition on $|x-4|$ that will assure that:
   (a) $|\sqrt{x}-2| < \frac{1}{2}$, (b) $|\sqrt{x}-2| < 10^{-2}$.
3. Let c be a cluster point of $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Prove that $\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to c} |f(x)-L| = 0$.
4. Let $f := \mathbb{R} \to \mathbb{R}$ and let $c \in \mathbb{R}$. Show that $\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to 0} f(x+c) = L$.
5. Let $I := (0, a)$ where $a > 0$, and let $g(x) := x^2$ for $x \in I$. For any points x, $c \in I$, show that $\left|g(x) - c^2\right| \le 2a|x-c|$. Use this inequality to prove that $\lim_{x\to c} x^2 = c^2$ for any $c \in I$.
6. Let I be an interval in $\mathbb{R}$, let $f : I \to \mathbb{R}$, and let $c \in I$. Suppose there exist constants K and L such that $|f(x) - L| \le K|x-c|$ for $x \in I$. Show that $\lim_{x\to c} f(x) = L$.
7. Show that $\lim_{x\to c} x^3 = c^3$ for any $c \in \mathbb{R}$.
8. Show that $\lim_{x\to c} \sqrt{x} = \sqrt{c}$ for any $c > 0$.
9. Use either the ε-δ definition of limit or the Sequential Criterion for limits, to establish the following limits.
   (a) $\lim_{x\to 2} \dfrac{1}{1-x} = -1$, (b) $\lim_{x\to 1} \dfrac{x}{1+x} = \dfrac{1}{2}$,
   (c) $\lim_{x\to 0} \dfrac{x^2}{|x|} = 0$, (d) $\lim_{x\to 1} \dfrac{x^2-x+1}{x+1} = \dfrac{1}{2}$.
10. Use the definition of limit to show that
    (a) $\lim_{x\to 2} \left(x^2+4x\right) = 12$, (b) $\lim_{x\to -1} \dfrac{x+5}{2x+3} = 4$.
11. Use the definition of limit to prove the following.
    (a) $\lim_{x\to 3} \dfrac{2x+3}{4x-9} = 3$, (b) $\lim_{x\to 6} \dfrac{x^2-3x}{x+3} = 2$.
12. Show that the following limits do *not* exist.
    (a) $\lim_{x\to 0} \dfrac{1}{x^2} \quad (x > 0)$, (b) $\lim_{x\to 0} \dfrac{1}{\sqrt{x}} \quad (x > 0)$,
    (c) $\lim_{x\to 0} (x + \operatorname{sgn}(x))$, (d) $\lim_{x\to 0} \sin\left(1/x^2\right)$.
13. Suppose the function $f : \mathbb{R} \to \mathbb{R}$ has limit L at 0, and let $a > 0$. If $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) := f(ax)$ for $x \in \mathbb{R}$, show that $\lim_{x\to 0} g(x) = L$.
14. Let $c \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x\to c} (f(x))^2 = L$.
    (a) Show that if $L = 0$, then $\lim_{x\to c} f(x) = 0$.
    (b) Show by example that if $L \neq 0$, then f may not have a limit at c.
15. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) := x$ if x is rational, and $f(x) = 0$ if x is irrational.
    (a) Show that f has a limit at $x = 0$.
    (b) Use a sequential argument to show that if $c \neq 0$, then f does not have a limit at c.
16. Let $f : \mathbb{R} \to \mathbb{R}$, let I be an *open* interval in $\mathbb{R}$, and let $c \in I$. If f_1 is the restriction of f to I, show that f_1 has a limit at c if and only if f has a limit at c, and that the limits are equal.
17. Let $f : \mathbb{R} \to \mathbb{R}$, let J be a *closed* interval in $\mathbb{R}$, and let $c \in J$. If f_2 is the restriction of f to J, show that if f has a limit at c then f_2 has a limit at c. Show by example that it does *not* follow that if f_2 has a limit at c, then f has a limit at c.